Energy dissipation in standing waves in rectangular basins

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(Received 29 November 1958)

The modulus of decay of standing waves of finite height is derived by assuming that the attenuation of the waves is due to viscous losses in boundary layers close to the solid walls. Dampings are observed in six basins of varying sizes. The basins are duplicated using glass and lucite for the wall materials. With liquids wetting the walls, the losses due to viscosity are slightly increased from causes presumably related to surface tension. With a liquid not wetting the walls (distilled water and lucite), losses from surface activity, of some obscure origin, outweigh many times the losses due to viscosity in the basins of smaller sizes. For moderately large basins, for which surface activity may be neglected, the agreement between the observed and computed rates of decay is found to be satisfactory.

1. Introduction

The aim of the present study is to examine the nature of energy dissipation in standing waves in rectangular basins. Observations of the rate of decay of standing waves reveal, upon proper analysis, that a significant amount of energy dissipation arises from interfacial effects as well as from the better known boundary-layer effects. This finding broadened the scope of the investigation beyond its original purpose, which was to determine how closely the observed rate of decay could be predicted by utilizing the boundary-layer concepts as put forth by Boussinesq (1898).

It has been found necessary to re-examine the boundary-layer analysis in order to obtain valid theoretical results. Briefly, in this analysis it is assumed that the entire loss of energy of waves is localized in the boundary layers adjacent to the solid walls. The liquid is at rest at the boundaries, and within the layers the motion is laminar. The limiting velocity at the outer edge of the layers is approximated by the velocity prevailing in the irrotational core. This is the velocity which would be attained at the walls had the layers been absent. The dissipation in the layers is assumed to be due solely to viscous effects associated with ordinary viscosity and velocity gradients. The equation giving the balance between the decrease of wave energy per cycle and the dissipation in the layers leads to the modulus of decay of wave height. The modulus is taken as the measure of the dissipation.

To examine the existence and the significance of any additional dissipation due to surface activities, basins of varying sizes were used. Also, the basins were constructed from two different materials, glass and lucite. The experiments with the glass basins were conducted using distilled water and glycerol aqueous solutions. These provided a variation in viscosities without changing markedly the surface tensions.

It has been found that for a given viscosity parameter, a type of Reynolds number to be defined later, dissipation increases as the size of the basin decreases. The increase is significant only for the basins of small size. On the other hand, this effect is reduced for ethyl alcohol, a liquid of smaller surface tension. All these facts may point to surface tension as a source of dissipation arising from the adherence of portions of liquids to the glass walls during the wave motion.

Some curious features were disclosed by the results obtained in the lucite basins using distilled water. Here the observed dissipation is twice or three times as large as that observed in glass basins. The greater increases are encountered in the smaller basins. Clean surfaces of lucite are not wetted by distilled water. When aerosol, a wetting agent, is added to the distilled water the dissipation is reduced, indeed to a degree that all effects become identical both in lucite and glass. Again, when using xylene or a mixture of xylene and a heavy mineral oil, which substances do wet the lucite and have also low surface tensions, dissipation is likewise reduced. All these facts suggest interfacial stresses of very large values as a source of dissipation, which differs depending on how the liquid adheres to the walls during the wave motion.

When the surface effects are abstracted, the observed and the computed values are brought into better agreement. The observed values after correction are, however, still about 10 % higher than the computed ones.

2. Dissipation of energy near solid walls

The modulus of decay is a measure of the energy losses realized during a complete oscillation. The number of oscillations n equals t/T, where T is the period of waves. One may take δn to represent the time for a single oscillation. Let δE be the increase of wave energy and ϵ the loss of energy from whatever source during a single complete oscillation, and thus

$$\frac{\delta E}{E} = -\frac{\epsilon \delta n}{E}. \tag{1}$$

Putting
$$\alpha = \frac{1}{2n}\int_0^n \frac{\epsilon}{E}\,\delta n, \tag{2}$$

equation (1) gives
$$\frac{E}{E_0} = e^{-2\alpha n}, \tag{3}$$

where E_0 is the wave energy at $t = 0$, or $n = 0$. Since E is proportional to the square of the semi-amplitudes a, one also has

$$\frac{a}{a_0} = e^{-\alpha n} = e^{-\alpha t/T}. \tag{4}$$

The quantity α, a dimensionless number, will be referred to as the 'modulus of decay'. From the definition implied in equation (2) it is seen that if ϵ/E be independent of n, hence independent of the amplitudes, the modulus of decay equals half the ratio of lost energy during a completed cycle to the energy of the wave during that cycle. Otherwise, α is half the average value of the corresponding ratios over the range n.

In the present experiments on the decay of standing waves, the observations are made on waves of finite amplitudes. As in these cases the end extreme depressions are smaller than the end extreme elevations, a question arises as to the expression of wave energy in terms of the end deflexions. This quantity is needed in the evaluation of the modulus of decay of the waves, equation (2). For this purpose it suffices to consider the second approximation to the surface heights.

The body of water of depth H is contained in a rectangular basin of length L and of width B, as indicated in figure 1. The free surface of the undisturbed water is taken as the (x, y)-plane and the axis of z is drawn upwards. The limiting planes are $x = 0$, $x = L$; $y = 0$, $y = B$; and $z = 0$, $z = -H$. The velocity components parallel to x, y, z are u, v, and w, respectively.

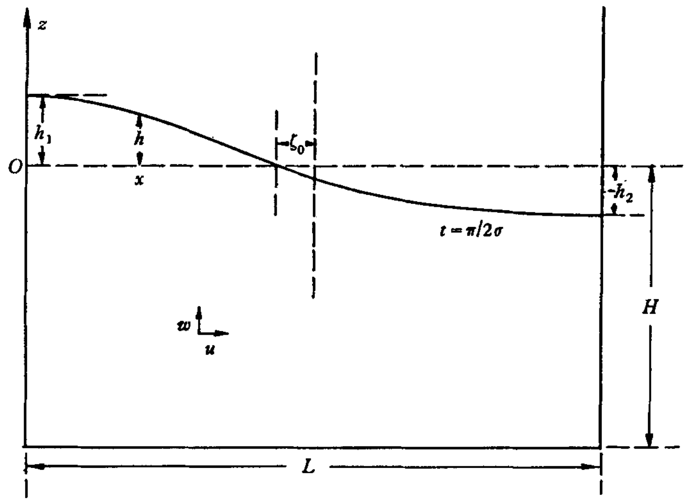

FIGURE 1. Notation diagram showing basin, standing wave, and axes.

In the second-order theory the velocity potential for the primary mode may be written as

$$\phi = A_1 \cosh k(z+H) \cos kx \cos \sigma t + A_2 \cosh 2k(z+H) \cos 2kx \sin 2\sigma t, \quad (5)$$

where

$$A_1 = -ag/\sigma \cosh kH, \quad A_2 = \tfrac{3}{16} A_1 ak/\sinh^3 kH, \quad \sigma^2 = gk \tanh kH,$$

with

$$k = \pi/L \quad \text{and} \quad \sigma = 2\pi/T.$$

Accordingly, the particle velocities in the main body of water are

$$u_0 = -\frac{gak}{\sigma} \frac{\cosh k(z+H)}{\cosh kH} \sin kx \cos \sigma t - \frac{3}{4} \frac{ga^2k^2}{\sigma} \frac{\cosh 2k(z+H)}{\sinh^2 kH \sinh 2kH} \sin 2kx \sin 2\sigma t, \quad (6)$$

and

$$w_0 = \frac{gak}{\sigma} \frac{\sinh k(z+H)}{\cosh kH} \cos kx \cos \sigma t + \frac{3}{4} \frac{ga^2k^2}{\sigma} \frac{\sinh 2k(z+H)}{\sinh^2 kH \sinh 2kH} \cos 2kx \sin 2\sigma t. \quad (7)$$

The surface elevation, when measured from the undisturbed water level, may be written as

$$h = a\cos kx \sin\sigma t + a(\tfrac{1}{4}ak)\,N_1\cos 2kx - a(\tfrac{1}{4}ak)\,N_2\cos 2kx\cos 2\sigma t, \tag{8}$$

where
$$N_1 = \frac{\cosh 2kH}{\sinh 2kH}, \quad N_2 = \frac{\cosh^2 kH(\cosh 2kH+2)}{\sinh^2 kH \sin 2kH}.$$

These equations are equivalent to the results of the second-order theory given by Miche (1944). The extreme values of h occur at the instant $t = nT + \pi/2\sigma$, and are thus

$$h_1 = a(1+\tfrac{1}{4}ak[N_1+N_2]) \quad (x=0),$$
$$h_2 = -a(1-\tfrac{1}{4}ak[N_1+N_2]) \quad (x=L).$$

Accordingly, the extreme end deflexions differ in absolute value, the elevations being larger than the depressions, and their ratio is

$$h_1/h_2 = -(1+\tfrac{1}{4}ak[N_1+N_2])/(1-\tfrac{1}{4}ak[N_1+N_2]). \tag{9}$$

At no time during the oscillations does the surface of the water in the basin assume a horizontal plane position. A residual displacement persisting at time $t=0$ has the value

$$h/a = \tfrac{1}{4}ak(N_1-N_2)\cos 2kx,$$

indicating, since N_2 is larger than N_1, the presence of a symmetrical hump of small elevation at the central portion of the basin.

A node in the ordinary sense does not exist. If, however, the node be defined as a point of the surface having zero displacement from the initial undisturbed level of water and the distance from the basin mid-section be denoted by $\zeta = \tfrac{1}{2}L - x$, the excursions of the node are given by

$$\sin k\zeta \sin\sigma t = \tfrac{1}{4}ak\cos 2k\zeta(N_1 - N_2\cos 2\sigma t).$$

Denote the maximum value of the excursions, which occurs at time $t = \pi/2\sigma$, by ζ_0. Thus

$$\sin k\zeta_0 = \tfrac{1}{4}ak(N_1+N_2)\cos 2k\zeta_0. \tag{10}$$

The energy E_1 of the wave per unit width of the basin may be expressed as

$$E_1 = \tfrac{1}{2}\rho g\int_0^{\frac{1}{2}\lambda} h^2\,dx + \tfrac{1}{2}\rho\int_0^{\frac{1}{2}\lambda}\phi_s\left[\left(\frac{\partial\phi}{\partial z}\right)_s - \left(\frac{\partial\phi}{\partial x}\right)_s\frac{\partial h}{\partial x}\right]dx,$$

of which the first integral in the right member represents the potential energy of the wave and the second the kinetic energy. The subscript s indicates that the quantities are evaluated for points on the surface. Introducing the values of ϕ and h from equations (5) and (8) and neglecting terms involving a^2k^2, the energy is

$$E_1 = \tfrac{1}{4}\pi\rho g a^2/k, \quad a = \tfrac{1}{2}(h_1-h_2), \tag{11}$$

which is the expression sought for.

Conceivably, the loss ϵ may be from different sources representing the sum

$$\epsilon = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_r, \tag{12}$$

and thus the modulus may be broken into several parts:

$$\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_r, \tag{13}$$

where
$$\alpha_r = \frac{1}{2n}\int_0^n \frac{\epsilon_r}{E}\,dn. \tag{14}$$

We shall denote the loss of energy per cycle arising from viscosity by $\epsilon_1 + \epsilon_1'$, of which the first represents the loss in the boundaries and the second that in the main body of liquid. First the derivation of ϵ_1 will be given.

Considering the vertical wall $y = 0$, and writing $z' = z + H$, the particle velocities in the layer are (following Boussinesq (1898))

$$u - u_0 = -f_1(x, z')\, e^{-\beta y} \cos(\sigma t - \beta y) \tag{15}$$

and

$$w - w_0 = -f_2(x, z')\, e^{-\beta y} \cos(\sigma t - \beta y), \tag{16}$$

where $\beta = \sqrt{(\sigma/2\nu)}$, u and w are the velocities within the layer, and u_0 and w_0 are the velocities outside the layer, that is $u_0 = f_1(x, z') \cos \sigma t$ and $w_0 = f_2(x, z') \cos \sigma t$. From equations (6) and (7) neglecting terms involving a^2k^2, one obtains

$$f_1(x, z') = -\frac{gak}{\sigma} \frac{\cosh kz'}{\cosh kH} \sin kx$$

and

$$f_2(x, z') = +\frac{gak}{\sigma} \frac{\sinh kz'}{\cosh kH} \cos kx.$$

At the bottom $z = -H$ (or $z' = 0$),

$$u - u_0 = -f_3(x)\, e^{-z'\beta} \cos(\sigma t - z'\beta), \tag{17}$$

where

$$f_3(x) = -\frac{gak}{\sigma} \frac{\sin kx}{\cosh kH}.$$

At the wall $x = 0$,

$$w - w_0 = -f_4(z')\, e^{-x\beta} \cos(\sigma t - x\beta), \tag{18}$$

where

$$f_4(z') = \frac{gak}{\sigma} \frac{\sinh kz'}{\cosh kH}.$$

Again, considering the vertical wall $y = 0$, the amount of energy dissipated in the viscous layer adjacent to the wall and during a complete cycle of period $T = 2\pi/\sigma$ is

$$\Delta E_1 = \mu \int_0^H \int_0^L \int_0^\infty \int_0^{2\pi/\sigma} \left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dt\, dy\, dx\, dz',$$

or, substituting from equations (15) and (16),

$$\Delta E_1 = \frac{g^2 a^2}{\sigma^2} \frac{\pi^2}{4} \sqrt{\left(\frac{\mu\rho}{2\sigma}\right)} \frac{\sinh 2kH}{\cosh^2 kH}. \tag{19}$$

The same amount of energy is dissipated also near the wall $y = B$. Near the wall $x = 0$, the corresponding loss of energy is given by

$$\Delta E_2 = \frac{g^2 a^2}{\sigma^2} \frac{\pi}{2} \sqrt{\left(\frac{\mu\rho}{2\sigma}\right)} \frac{Bk}{\cosh^2 kH} \left[\frac{\sinh 2kH}{2} - kH\right]. \tag{20}$$

The same amount of energy is dissipated also near the wall $x = L$.

Near the bottom, $z = -H$ or $z' = 0$, the corresponding loss is given by

$$\Delta E_3 = \frac{g^2 a^2}{\sigma^2} \frac{\pi^2}{2} \sqrt{\left(\frac{\mu\rho}{2\sigma}\right)} \frac{Bk}{\cosh^2 kH}. \tag{21}$$

The total amount of energy lost during a completed cycle, ϵ_1, is now

$$\epsilon_1 = 2\Delta E_1 + 2\Delta E_2 + \Delta E_3,$$

and from equations (19), (20), and (21)

$$\epsilon_1 = \frac{g^2a^2}{\sigma^2}\sqrt{\left(\frac{\mu\rho}{2\sigma}\right)}\,\pi \tanh kH\chi,$$

or, using equation (5),

$$\epsilon_1 = \frac{\pi g a^2}{k}\sqrt{\left(\frac{\mu\rho}{2\sigma}\right)}\,\chi, \tag{22}$$

where $\chi = (\pi + Bk) + Bk(\pi - 2kH)/\sinh 2kH.$

Equations (11) and (22) give for the modulus of decay, $\alpha_1 = \epsilon_1/2E_1$, the value

$$\alpha_1 = \pi^{-\frac{1}{2}}\chi\nu^{\frac{1}{2}}T^{\frac{1}{2}}/B. \tag{23}$$

The modulus varies directly with the number $\nu^{\frac{1}{2}}T^{\frac{1}{2}}/B$, which is the viscosity parameter, a form of the Reynolds number. For basins and waves similar in shape the modulus varies directly with the viscosity parameter.

When the loss per cycle in the main body due to viscosity, ϵ_1', is computed by the method of Lamb (1932, § 348) applied to the case of progressive surface waves, it proves to be $\epsilon_1' = \mu\pi a^2 gkBT$. Hence, the corresponding modulus $\alpha_1' = \epsilon_1'/2E_1$ is

$$\alpha_1' = 2k^2B^2\nu T/B^2. \tag{24}$$

This part of the modulus varies as the square of the viscosity parameter.

3. Experiments on the characteristics of waves

The study of the damping was made with various basins of geometrically similar construction but of different sizes. The dimensions of these together with the water depths are given in table 1. In the basic dimensions the experimental waves were similar; the depth ratio $H/L = 0{\cdot}425$, width ratio $B/L = 0{\cdot}217$, and $kH = 1{\cdot}335$.

Basin	H (cm)	L (cm)	B (cm)	H/L	B/L	T (sec)
A	103·0	242·0	52·6	0·425	0·217	1·89
B	40·4	94·7	20·6	0·426	0·217	1·20
C	20·1	47·4	10·1	0·424	0·217	0·83
D	15·7	37·0	8·0	0·425	0·216	0·73
E	11·8	27·8	6·1	0·425	0·218	0·64
F	10·1	23·8	5·2	0·425	0·218	0·59

TABLE 1. Dimensions of basins

All the boundaries of each basin, the side walls, the end walls, and the bottom were smooth. Two sets of basins were constructed, one of glass and the other of lucite. The walls were transparent, and thus the surface could be viewed either from the sides or from the ends.

The essential details of the apparatus are shown in figure 2. The waves are produced by rocking the basin on a transverse axle placed at the mid-bottom. The cam is actuated by a variable speed d.c. motor. The eccentricity of the cam determines the amplitude of the rocking motion. At the resonance frequencies the waves are readily built up to large amplitudes even when the eccentricities of the cam are small. After the creation of a wave of the desired height the rocking is stopped, care being taken to immobilize the basin during subsequent surges. Observations may then be made of the deflexion of the ends or the excursions of the nodal point or the gradual attenuation of the wave height.

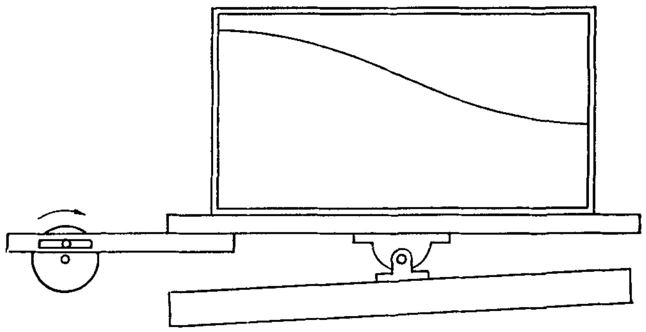

FIGURE 2. Sketch of rocking basin apparatus.

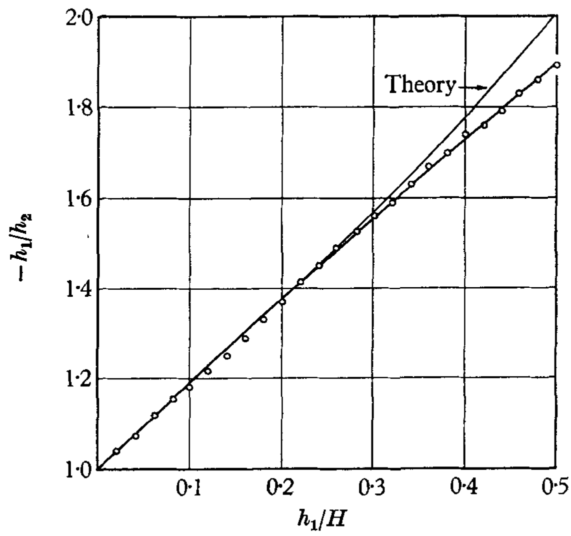

FIGURE 3. Variation in the ratio of the extreme maximum end deflexions.

To establish the range of wave heights within which the result of the second-order theory as derived above is applicable, the examination is confined to the end deflexions and to the maximum nodal excursions. The results of observation on the end deflexions are given in figure 3 where the ratio $-h_1/h_2$ is plotted against h_1/H as shown by the circles. These are average values determined from three basins, the two larger ones and the smallest. The theoretical curve is computed from equation (9). As long as h_1/H, that is the ratio of maximum end deflexion to the depth of water, is less than $\frac{1}{3}$, the ratio of the end deflexions is correctly given by the theory. Above this limit, that is $h_1/H > \frac{1}{3}$, the observed values fall below the theoretical ones.

It is instructive to consider also the dependence of the extreme nodal excursions on the wave height. Representative observations are plotted in figure 4. In this case the width B was 20 cm, and the water surface lines were photographed at the times of maximum deflexions. The photographs also showed the images of the lines traced on the side wall giving the position of the undisturbed water line and the mid-section plane. The desired data were obtained from these photographs. The theoretical curve was computed from equation (10) with $kH = 1\cdot335$. For the wave height ratios a/H smaller than 0·3, the agreement between theory and observation is satisfactory. For ratios greater than this the observed nodal excursions are smaller than the theoretical estimates.

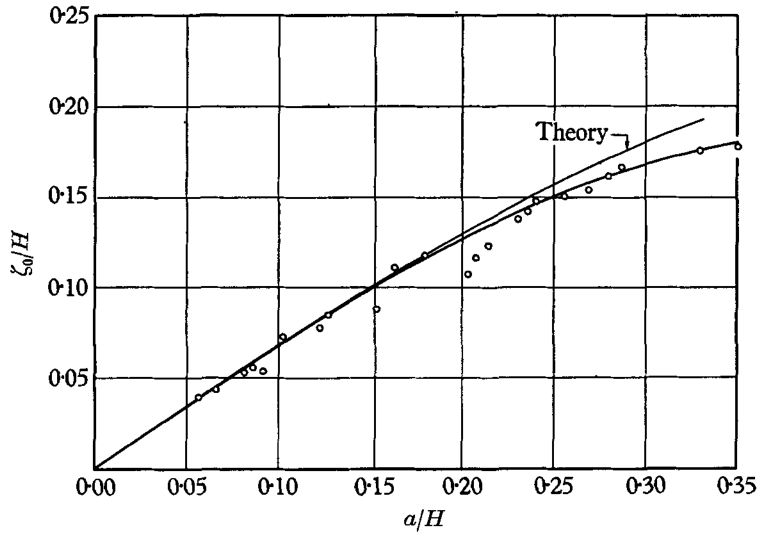

FIGURE 4. Variation in the extreme excursions of the node.

It is inferred that the second-order theory as given above is sufficiently accurate when the deflexion ratio h_1/H is less than $\frac{1}{3}$.

4. Energy dissipation in glass basins

The determination of rate of decay of the waves from observation of successive end deflexions was carried out in two different ways, depending on the size of the basin. For the larger basins visual readings were made simultaneously by two observers, while for the smaller basins the information was obtained by photographing the wave surface with a motion-picture camera and later examining individual frames for the maximum end positions. On the basis of information given in figure 3, the value $-h_2$ corresponding to a given h_1 is ascertained, and the quantity $a = \frac{1}{2}(h_1 - h_2)$, the semi-wave height, is formed. Finally, the curve giving this a as a function of t is established.

To show that the observed dampings of standing waves in these experiments are characterized by a modulus of decay α as required by equation (4) and with the amplitude as $a = \frac{1}{2}(h_1 - h_2)$, the data with water from the three channels A, B and C are given in figure 5. The plotting is semi-logarithmic and a/a_0 is plotted against t/T. In these data, a_0 is the initial value, $a_0 = 0{\cdot}25H$. The alinements of the observed points are linear and a constant value of the modulus of decay is assured in each case.

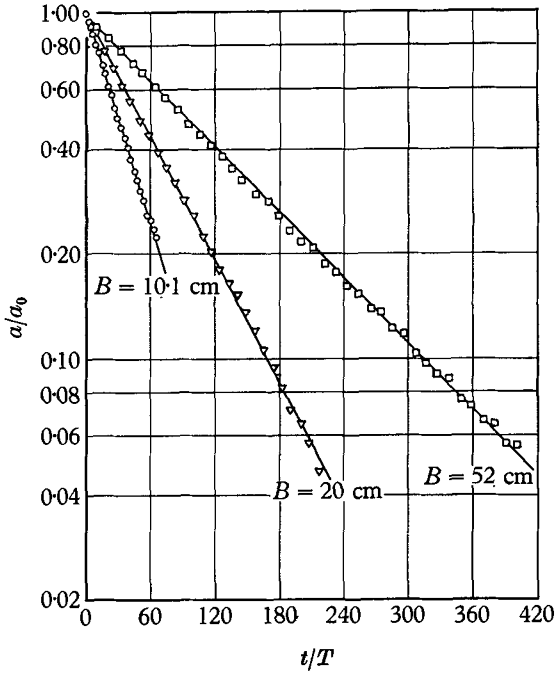

FIGURE 5. Examples of decay of amplitude of oscillations with time.

The modulus of decay in tests considered later is not obtained from curves similar to the ones shown in figure 5, but rather, using equation (4), from the following relation:

$$\alpha = \frac{T}{t_r}\log\left(\frac{a_0}{a_r}\right), \tag{25}$$

where a_0 is an initial value and a_r a subsequent value, inferior to a_0, and t_r is the time for the semi-amplitudes of the oscillations to decrease from a_0 to a_r. Since the computed values of the period T of oscillations from equation (5) are in agreement with observation, the theoretical values are used. Whenever a curve $a = a(t)$ is available, the quantity t_r is read from the curve, after a_0 and a_r are selected. For reliability of the results concerning the modulus of decay, the latter is computed several times in a given test by selecting various values of a_r in decreasing order while keeping a_0 the same.

To determine the effect of viscosity on the modulus of decay, and thus indirectly on the rates of energy dissipation, distilled water and glycerol aqueous solutions were first chosen. This choice of the liquids provided a considerable variation in the viscosity with only insignificant changes in surface tension. To be assured of the right values of the viscosity, separate determinations were made of the kinematic viscosity for each concentration at the temperature applying to test condition by a modified Oswalt type viscosimeter.

The effect of viscosity was examined in all of the basins except the largest, with $B = 52$ cm, where tests were confined to city drinking water. The observed moduli of decay are presented in figure 6, where it is seen that the values depend on the width of the basin B as well as on the viscosity parameter. This separate dependence on B would not be expected considering the fact that the waves were

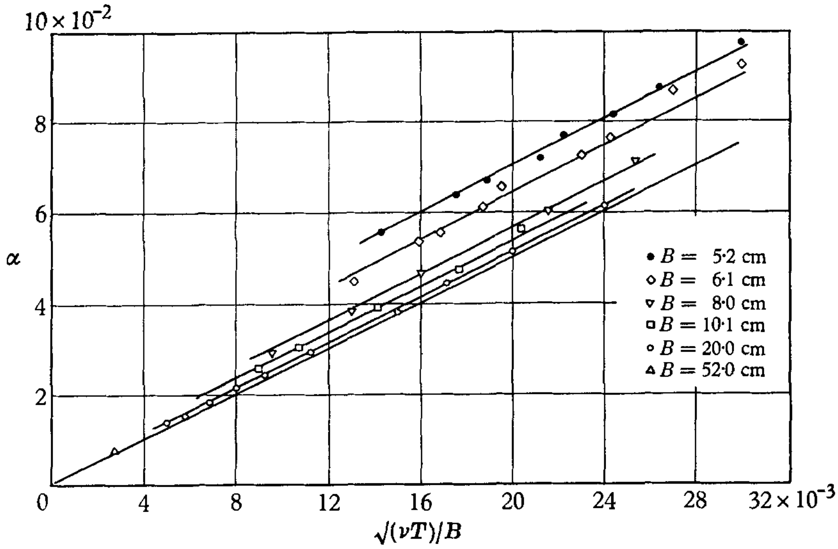

FIGURE 6. Observed modulus of decay of amplitudes in glass basins

B	$\nu^{\frac{1}{2}}T^{\frac{1}{2}}/B$	Glycerol-water (α)	Alcohol (α)	$\gamma = 0$ (α_1)
5·2	$19{\cdot}6\times10^{-3}$	0·0695	0·0552	0·0495
6·1	$18{\cdot}1\times10^{-3}$	0·0605	0·0495	0·0458
8·0	$15{\cdot}3\times10^{-3}$	0·0445	0·0397	0·0375
10·1	$13{\cdot}6\times10^{-3}$	0·0380	0·0349	0·0335

TABLE 2. Effect of surface tension on modulus of decay

similar in depth and length in terms of B. This anomaly indicated a possible effect arising from some obscure surface activity phenomenon. Supposing that this might depend on surface tension, tests were made with ethyl alcohol, which has a much lower surface tension than the solutions previously used. A comparison is made between the results with alcohol and with glycerol aqueous solution in table 2 for equal viscosity parameters. It is obvious that alcohol gives smaller dissipation.

One may break the observed modulus of decay α into two parts, $\alpha = \alpha_1 + \alpha_2$, α_1 being the part due to viscosity and α_2 due to surface tension. On the basis of dimensional reasoning for similar waves, that is with the ratios B/H and kH invariable, one may deduce that

$$\alpha_1 = \alpha_1(\nu^{\frac{1}{2}}T^{\frac{1}{2}}/B), \tag{26}$$

and

$$\alpha_2 = \alpha_2(\gamma T^2/\rho B^3). \tag{27}$$

Here γ is the surface tension of liquid in contact with air. The form of α_1 has been determined theoretically as equation (23). As regards α_2, an empirical determination will be attempted.

In figure 6 the straight line passing through the origin and with a slope approximating the slopes of the remaining straight lines drawn through the observed points from the various basins is considered as the limiting line. It represents α_1. The quantity α_2 is read as the offset. The dependence of α_2, thus determined, on the surface parameter is shown in figure 7. The points represented by the circles

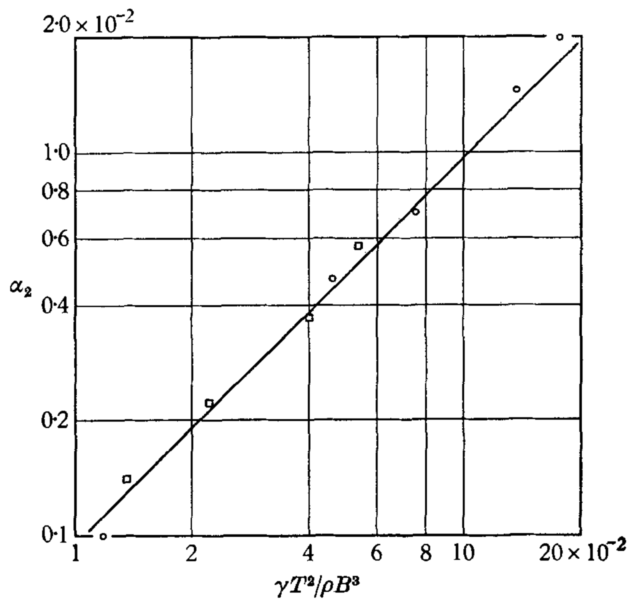

FIGURE 7. Effect of surface tension on the modulus of decay.

are from tests of the glycerol aqueous solutions and the squares, from alcohol. The results from the two groups are in agreement and suggest that the conjecture that surface tension is involved in the dissipation may be a valid one. The line in figure 7 corresponds to

$$\alpha_2 = 0{\cdot}10\frac{\gamma T^2}{\rho B^3}. \tag{28}$$

5. Discussion of energy dissipation due to viscosity

Presumably, if the part of the modulus of decay a_2 computed on the basis of the empirical relation (28) be subtracted from the observed modulus of decay α, the remaining part α_1 is due to viscosity, assuming that no other causes of dissipation are present. The experimental values of α_1, thus determined from the tests with the glass basins, are shown in figure 8. Introducing numerical values in equation (23), which purports to give the modulus of decay as relating to the boundary layers close to the solid surfaces, there results

$$\alpha_1 = 2{\cdot}16(\nu T)^{\frac{1}{2}}/B, \tag{29}$$

the graph of which is the lowest curve in figure 8. Between the two extremes there is a disparity of the amount

$$(\alpha_1^0 - \alpha_1)/\alpha_1 = 6{\cdot}2(\nu T)^{\frac{1}{2}}/B, \tag{30}$$

where the superscript 0 refers to the experimental curve.

It is not necessary that this disparity be ascribed to observational difficulties or to uncertainties of analysis. For two other items may still be considered. One is in regard to the reduction of channel width owing to the boundary layers. Let this reduction be 2Δ, and the corresponding change in the modulus be $\Delta\alpha_1$. From equation (23) it follows if B/L be small, that $\Delta\alpha_1/\alpha_1 = 2\Delta/B$. It might be said for

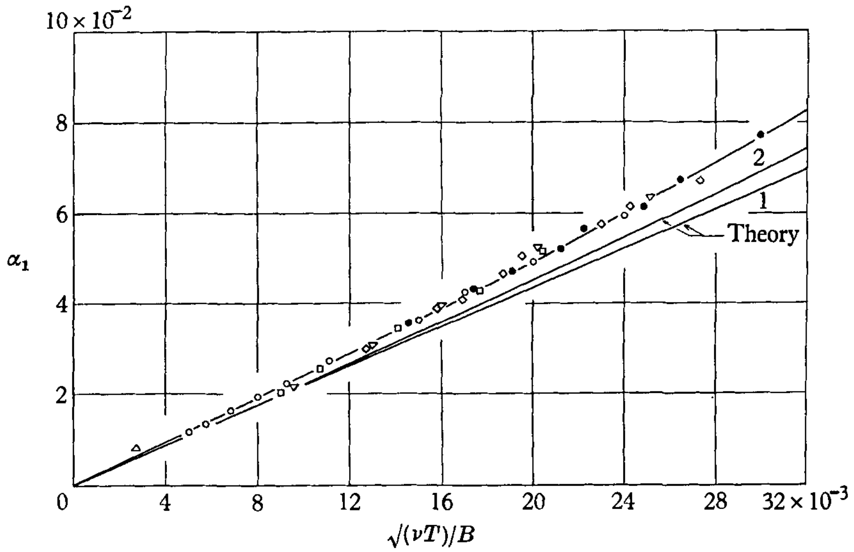

FIGURE 8. Modulus of decay due to viscosity in glass basins. Glycerol solutions

the purpose of estimation that nearly all of the kinetic energy of the layer of thickness

$$\delta = 2\pi(2\nu/\sigma)^{\frac{1}{2}} \tag{31}$$

is concentrated in the portion of the width $\frac{3}{4}\delta$, and the layer of negligible energy is $\frac{1}{4}\delta$. Identifying Δ with $\frac{1}{4}\delta$, one has $2\Delta = \pi(2\nu/\sigma)^{\frac{1}{2}}$ and hence

$$\Delta\alpha_1/\alpha_1 = \pi^{\frac{1}{2}}(\nu T)^{\frac{1}{2}}/B. \tag{32}$$

The second theoretical curve of figure 8 is evaluated with this correction added.

In view of the change the disparity now between the observed and the computed modulus is about

$$(\alpha_1^0 - \alpha_1 - \Delta\alpha_1)/\alpha_1 = 4{\cdot}5(\nu T)^{\frac{1}{2}}/B. \tag{33}$$

If it be assumed that there is dissipation in the body of water due to internal eddy agitation, then the last written relation may be used to obtain the eddy viscosity of this agitation. If one may assign the value $m\nu$ to the eddy viscosity, then equation (24), which purports to give the part of modulus of decay due to internal dissipation, yields (taking α_1 from equation (29))

$$\alpha_1'/\alpha_1 = 0{\cdot}43\, m(\nu T)^{\frac{1}{2}}/B. \tag{34}$$

Comparing this with equation (33), since $\alpha_1' = (\alpha_1^0 - \alpha_1 - \Delta\alpha_1)$, it is seen that the eddy viscosity of the internal agitation is about 10 times the natural viscosity, a plausible value.

For the purpose of ascertaining if an internal agitation is actually present, the following observation was made in the basin with $B = 20$ cm and $H = 40$ cm. While the water in the basin was still, a saturated solution of potassium permanganate was introduced by means of narrow tubes and allowed to form a layer 2 mm thick and spread evenly over the entire area of the bottom. In the first few oscillations, portions of coloured liquid were seen to rise from the bottom in the proximity of the ends. The initial rise amounted to about 6 cm, and subsequently the coloured portions were seen to move towards the basin mid-section. During this motion the areas of coloured portions grew in size. As measured from the photographs taken at 10 sec intervals, the sequence of the sizes corresponding to these successive times proved to be 42, 160, 520 and 670 cm^2. There were circulations in the two cells of the basin and also very marked dispersions. Because of the ratio of wave height to boundary layer thickness being very large, it was not possible to relate the observations to the analytical results of Longuet-Higgins (1953) on mass transportation in standing waves.

The elementary theory of boundary layers does give adequately the major portion of viscous losses in the standing waves of the smooth-walled basins. The same method may be used with confidence to evaluate similar losses for progressive waves in laboratory studies relating to models.

6. Anomalous mode of decay in lucite basins

The modes of decay of oscillations in lucite basins using distilled water surprisingly enough prove to be different. Dissipation is much greater in comparison to that with a glass basin. The difference becomes more pronounced with decreasing basin sizes. To afford a more detailed comparison, the results with the

B (cm)	$\nu^{\frac{1}{2}}T^{\frac{1}{2}}/B$	Lucite (α)	Glass (α)
20·0	0·00530	0·0224	0·0164
10·1	0·00886	0·0497	0·0245
8·0	0·01021	0·0624	0·0290
6·1	0·01323	0·1165	0·0398
5·2	0·01461	0·1382	0·0451

TABLE 3. Modulus of decay in lucite and glass basins (distilled water).

lucite channel are entered in table 3. Another point worthy of mention is the fact that with lucite the modulus of decay is not a constant but increases with decreasing amplitudes. To illustrate the point, the results from the three smaller channels are collected in table 4. Here α is the average of the modulus for the first n oscillations. The average value increases with n, indicating that the ratio of lost energy during a cycle to energy of wave increases very markedly as the amplitudes of oscillations become smaller.

To render a graphic comparison between lucite and glass, the data of table 3 are reproduced in figure 9. Of the three curves, the upper one refers to the lucite results for large amplitudes of oscillations. The remaining two refer to the glass results, the lower curve representing purely viscous effects. Here again with the lucite results the observed modulus of decay may be broken into two parts, α_1 and α_2, α_1 representing the part due solely to viscous actions and α_2 due to a surface

B (cm)	$\nu^{\frac{1}{2}}T^{\frac{1}{2}}/B$	a_0/a	n	α
8·0	0·01028	4·28	23	0·0632
		5·46	26	0·0632
		6·90	28	0·0690
		7·97	29	0·0716
		10·77	31	0·0766
		14·23	33	0·0812
6·1	0·01331	6·53	16	0·1173
		8·30	17	0·1245
		10·41	18	0·1301
		13·05	19	0·1352
		16·18	20	0·1392
		23·39	22	0·1433
5·2	0·01463	5·21	11	0·1501
		6·79	12	0·1596
		9·60	13	0·1740
		15·15	14	0·1941

TABLE 4. Variation of modulus of decay with amplitude (distilled water in lucite basins)

activity of some kind. Effecting the separations, the resulting α_2 is plotted in figure 10 against the surface tension number based on the surface tension of distilled water. The distribution of the points may for simplicity be represented by the straight line

$$\alpha_2 = 0\cdot6\frac{\gamma T^2}{\rho B^3}. \tag{35}$$

Comparing this with equation (28), it is inferred that the loss of energy from surface activity of distilled water in contact with lucite, is about six times as large as when distilled water is in contact with glass.

Since distilled water wets glass but does not wet lucite, the latter fact is regarded as the only cause of the anomalous and large decay conditions in the lucite basin tests. To put this explanation to task, a parallel series of experiments were conducted, with glass and with lucite basins having $B = 10\cdot0$ cm, starting with distilled water and then progressively adding amounts of aerosol. Aerosol, a wetting agent, causes the water to adhere to lucite. The tests were repeated three times to test the reliability of this procedure. The average results are shown in figure 11, where the modulus of decay from larger amplitudes are plotted against the concentration of aerosol. With the additions of aerosol, the loss of energy of the oscillation in the lucite channel decreases gradually, and finally no difference can be seen between the results in the two basins. During the addition of aerosol,

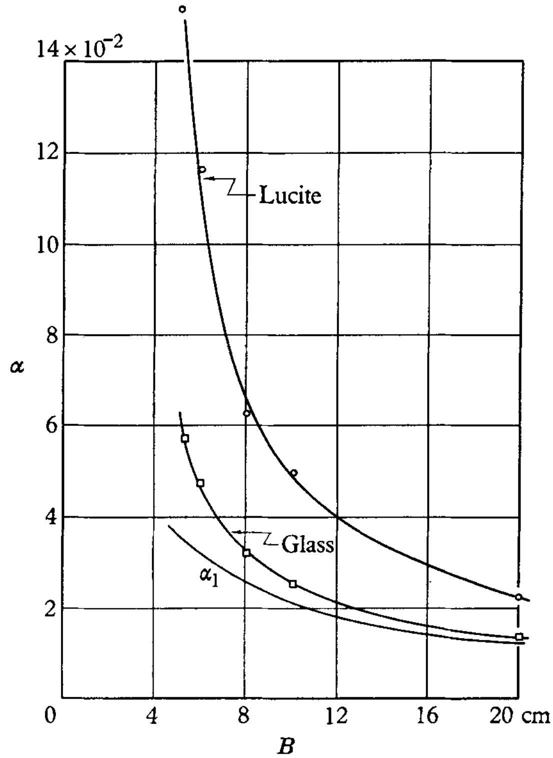

FIGURE 9. Modulus of decay in glass and lucite basins. Distilled water.

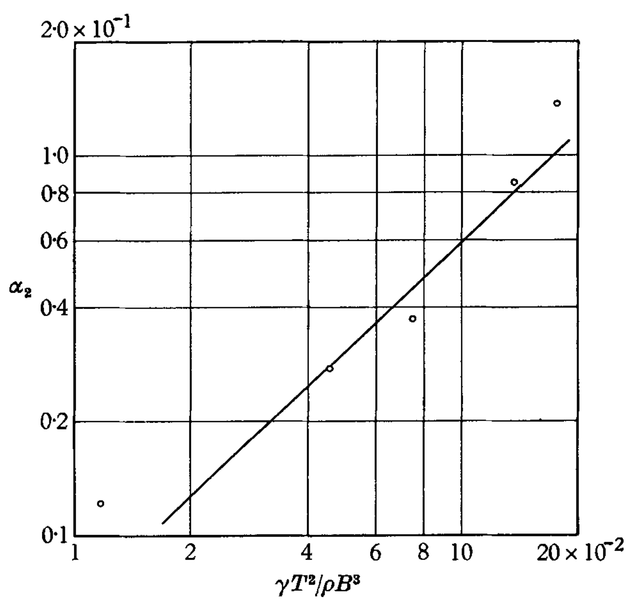

FIGURE 10. Surface activity effect in lucite basins.

the surface tension of the liquids was determined. The specific tension as a function of aerosol concentration is shown in figure 12. Referring once more to figure 11, it will be noted that a very slight addition of aerosol to distilled water in a glass basin causes a marked increase in the losses. For this no explanation can be given. However, here again the losses in general decrease with the aerosol concentration, and the decrease is presumably due to the lower surface tension.

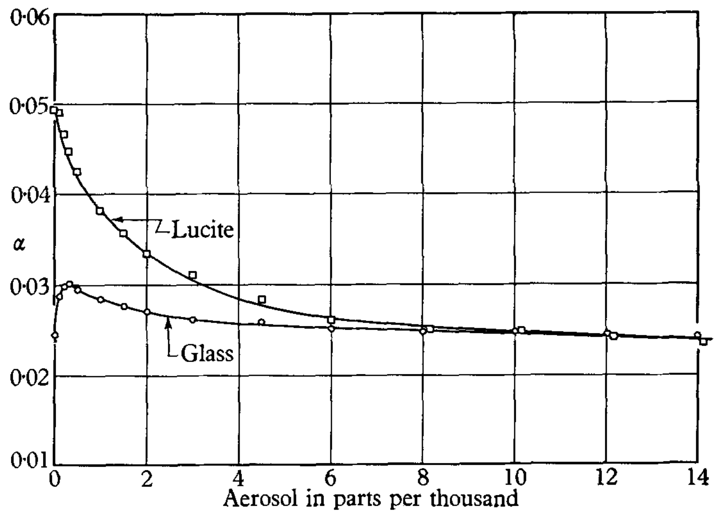

FIGURE 11. Effect of aerosol in distilled water in lucite and glass basins.

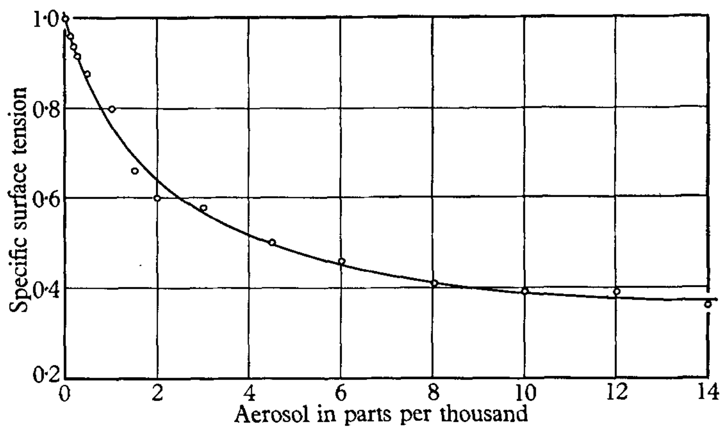

FIGURE 12. Surface tension of water charged with aerosol.

To demonstrate further that, with liquids naturally wetting the lucite, increased losses are not realized, additional tests were conducted with the four smaller lucite basins employing xylene and a mixture of 25 % xylene and marcus oil. These liquids readily spread over lucite and have low surface tensions. The resulting values of the modulus of decay from these runs after subtracting the part α_2 due to surface tension, this part computed using the empirical result of equation (28), are reproduced in figure 13. The curve drawn through the points reproduces almost identically the similar curve obtained with glass basins.

Possibly the large additional losses of energy, in the cases in which liquids do not wet the material of the walls, are due mainly to interfacial tension. This comes into play during the surging actions of the oscillations and, alternately, at the times when the liquid renews its contact with the walls. The resulting effects are more severe when basins are small. In fact in small basins the appearance of the liquid surfaces during oscillations are rough, irregular and sometimes are covered with ripples and small waves. In contrast, when the liquids wet the walls and surface tension is small, liquid surfaces are smooth and have a glossy appearance. It is also conceivable that added losses result from the hydrodynamic contact conditions in the interior wall surfaces between liquids and the walls that are not wetted by the liquid. To confirm or to negate the latter possibility and in general

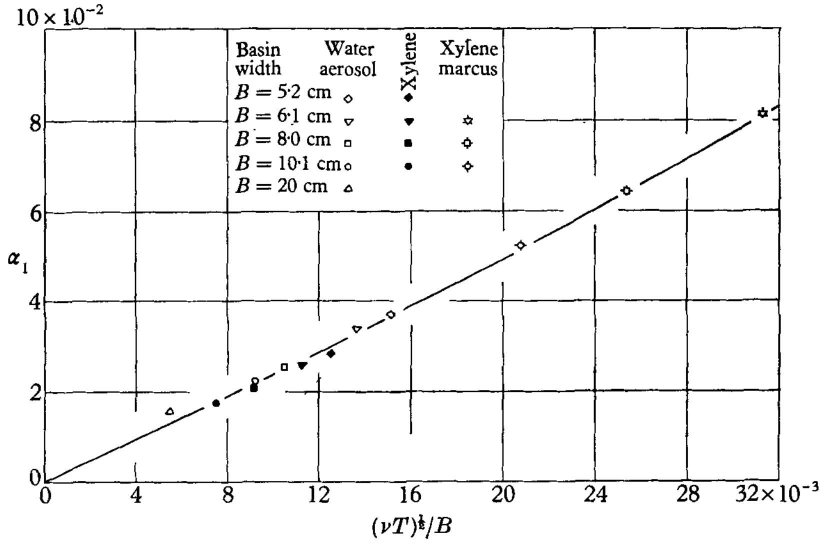

FIGURE 13. Modulus of decay due to viscosity in lucite basins. Wetting liquids.

to offer a satisfactory explanation of questions considered in this section, the study has been extended to the damped oscillations of circular disks immersed in liquids, to the vertical oscillations of plates and polished brass rods partially immersed, and the direct measurement of meniscus forces on partially immersed plates during a given cycle. It appears that the data of the new tests would be suitable for the evaluation of the dissipation from a meniscus. The need for such a study was first emphasized by Benjamin & Ursell (1954). Using a 5·4 cm Perspex tube for the experiments to confirm their theory of stability of plane surfaces, they found a rate of dissipation twenty times the value calculated from viscous dissipation. This is in line with the above expressed idea that the dissipation from meniscus becomes more pronounced when the vessel of a hydrophobic material containing the liquid is made smaller. In the experiments of Case & Parkinson (1957) a somewhat different situation is encountered. By polishing the inside surfaces of brass cylinders, the damping rates of the surface waves attained the values calculated from viscous dissipation. As the corresponding losses for the unpolished cylinders were 2 or 3 times as large, a question is posed as to the effects of very small roughness on damping. It would seem that the desired explanation should be such as to relate the state of metal surface to the meniscus deformation.

This investigation has been sponsored by the Office of Naval Research. The author acknowledges gratefully the helpful suggestions of Dr Galen B. Schubauer, Chief of the Fluid Mechanics Section of the National Bureau of Standards, and of Dr Robert Stoneley, Department of Geodesy and Geophysics, University of Cambridge, and the careful experimentation of his assistants, Mr Marion R. Brockman and Mr Victor Brame, Jr.

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